International Journal of Theoretical Physics, Vol. 16, No. 9 (1977), pp. 663–670

The Real Geometry of Complex Space-Times

NICHOLAS WOODHOUSE

The Mathematical Institute, 24–29 St. Giles, Oxford

Received: 1 *June* 1977

Abstract

This paper describes the geometry of complex space-times from the real point of view and presents some miscellaneous results on the existence and nature of real slices.

1. *Introduction*

In this paper, I shall collect together some facts concerning the real geometry of complex Riemannian manifolds, and, in particular, I shall discuss the question of the existence of real slices in complexified space-times. This question is important in two contexts:

(1) In quantum field theory, it is sometimes helpful analytically to continue objects such as propagators and formal path integral expressions from space-time to a positive definite Riemannian manifold, thought of as a real slice of complex space-time. For example, this can be done in Minkowski and de Sitter space-times, and in the Schwarzschild solution (Hartle and Hawking, 1976; Hawking and Gibbons, 1977).

(2) In general the complex versions of Einstein's equations are easier to solve than their real forms (for example, see Plebanski and Robinson, 1976, and references cited therein). But, having found a complex solution

$$ds^2 = g_{ab} dz^a dz^b \tag{1.1}$$

(where the metric coefficients g_{ab} are holomorphic functions of the coordinates), one is faced with the problem of finding a real form of the metric, given by transforming to new coordinates $w^a = w^a(z^b)$ and restricting the w^a's to real values. Only exceptional choices of new coordinates will lead to a real metric.

I shall begin by recalling some elementary ideas about the relationship between real and complex manifolds (for more detail, see Kobayashi and Nomizu, 1969).

2. *Real and Complex Manifolds*

(1) An n-dimensional complex manifold can also be thought of as a $2n$-dimensional real manifold.

(2) From an n-dimensional real analytic manifold, it is possible to construct n-dimensional complex manifolds by complexification.

In the first case, one simply takes the real and imaginary parts x^a and y^a of a system of complex analytic coordinates $z^a = x^a + iy^a$ as real coordinates. This gives a $2n$-dimensional real analytic manifold M with a complex structure J in the tangent space at each point. That is, J is a real tensor field of type $\binom{1}{1}$ (one covariant and one contravariant index) satisfying

$$J^2 = -1 \qquad \text{or } J_\alpha{}^\beta J_\beta{}^\gamma = -\delta_\alpha{}^\gamma \tag{2.1}$$

(Greek indices run over $1, 2, \ldots, n$). In the coordinates $\{x^a, y^a\}$, J is given by

$$J\left(\frac{\partial}{\partial x^a}\right) = \frac{\partial}{\partial y^a} \quad \text{and} \quad J\left(\frac{\partial}{\partial y^a}\right) = -\frac{\partial}{\partial x^a} \tag{2.2}$$

(Roman indices run over $1, 2, \ldots, n$).

A real manifold equipped with such a tensor field (that is, with a complex structure in the tangent space at each point) is called an *almost complex manifold.* A necessary and sufficient condition that an almost complex manifold should be a complex manifold is that the torsion tensor S of J should vanish; S is the $\binom{1}{2}$ tensor field defined by

$$S(X, Y) = 2([JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]) \tag{2.3}$$

where X and Y are vector fields. In coordinates,

$$S^\alpha_{\beta\gamma} = 4(J_\delta{}^\alpha J^\delta_{[\beta,\gamma]} + J^\delta_{[\beta} J^\alpha_{\gamma],\delta}) \tag{2.4}$$

When S vanishes, J is said to be *integrable.* The sufficiency of this condition is fairly easy to establish if it is assumed that M and J are real analytic, but with less stringent differentiability requirements the proof is much harder (see Kobayashi and Nomizu, 1969, p. 124).

In the second case, starting with an n-dimensional real analytic manifold N, one constructs an n-dimensional complex manifold by allowing the coordinates to take on complex values and by analytically continuing the coordinate transformation maps. The resulting complex manifold M is also a $2n$-dimensional real analytic manifold in which N sits as an n-dimensional submanifold, given locally by

$$z^a = \bar{z}^a \text{ or } y^a = 0 \tag{2.5}$$

in the analytically continued coordinates $z^a = x^a + iy^a$.

In general, this process is not unique and it only makes sense as defining a slight thickening of N into the complex (Shutrick, 1958).

Conversely, given an n-dimensional complex manifold M, a real slice N of M is defined to be an n-dimensional real submanifold of M given locally by

$z^a = \bar{z}^a$ in some system of complex analytic coordinates. In real terms, N is an n-dimensional submanifold of M (now thought of as a $2n$-dimensional real manifold) with the properties

(1) N is real analytic

(2) $\forall n \in N, T_nM = T_nN \oplus J(T_nN)$ [That is, N is *totally real.* Equivalently, if X is tangent to N then JX is not tangent to N. That is, N has no $(1, 0)$ tangents[1].]

3. *Complex Riemannian Manifolds*

A complex Riemannian manifold or complex space-time is a complex manifold M (usually four-dimensional) together with a holomorphic metric g. In local complex coordinates z^a,

$$g = g_{ab}(z)dz^a \otimes dz^b \tag{3.1}$$

with the coefficients g_{ab} holomorphic functions of the coordinates z^a.

In real terms, g is a complex-valued covariant tensor field on M which is

(1) symmetric: $g(X, Y) = g(Y, X)$ for all vectors X and Y

(2) of type $(2,0)$: $g(X, Y) + ig(JX, Y) = 0\ \forall\ X, Y$, that is, g annihilates all $(0, 1)$ vectors

(3) nondegenerate: $g(X, Y) - ig(JX, Y) = 0\ \forall\ Y$ only if $X = 0$

(Note that all complex metrics have the same signature.) The tensor field g can be split into its real and imaginary parts: $g = h + ik$, where h and k are real covariant tensor fields on M (as a real manifold). Translating (1)-(3) into statements about h, k and the integrable almost complex structure J we have the following:

(1′) h and k are symmetric: $h(X, Y) = h(Y, X)$ and $k(X, Y) = k(Y, X)$ for all vectors X and Y.

(2′) $h(X, Y) = k(JX, Y)$ and $k(X, Y) = -h(JX, Y)\ \forall\ X, Y$.

(3′) h and k are nondegenerate.

It follows from (2′) that any two of h, k and J determine the third. Also from (2′) we obtain the compatibility conditions

$$h(JX, JY) = -h(X, Y) \text{ and } k(JX, JY) = -k(X, Y)\ \forall X, Y \tag{3.2}$$

These imply that the real metrics h and k have signatures $+, +, +, +, -, -, -, -$ (since J interchanges spacelike and timelike vectors for both h and k).

Examples.

(1) $$\begin{aligned} ds^2 &= (dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 \\ &= [(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 - (dy^1)^2 - (dy^2)^2 \\ &\quad -(dy^3)^2 - (dy^4)^2] + i[2(dx^1dy^1 + dx^2dy^2 + dx^3dy^3 + dx^4dy^4)] \end{aligned}$$

[1] A $(1, 0)$ vector X is a complex vector satisfying $JX = iX$. Similarly, a $(0, 1)$ vector satisfies $JX = -iX$.

The metric in the first square bracket is h, that in the second is k.

(2) (One complex dimension): If $g = f(z)dz^2$, where $f = u + iv$, then

$$h_{\alpha\beta} = \begin{bmatrix} u & -v \\ -v & -u \end{bmatrix} \text{ and } k_{\alpha\beta} = \begin{bmatrix} v & u \\ u & -v \end{bmatrix}$$

4. *Connections*

The holomorphic metric g defines a complex connection, denoted D: the corresponding Christoffel symbols are constructed from g in the same way as in a real space (Flaherty, 1976). In coordinate-free notation, let $\alpha = \alpha_a dz^a$ be a holomorphic 1-form. Then the covariant derivative $D\alpha = (D_a\alpha_b)dz^a \otimes dz^b$ of α is defined by

$$D\alpha = \tfrac{1}{2}\mathscr{L}_A(g) + d\alpha \tag{4.1}$$

where A is the holomorphic vector field given by $A^a g_{ab} = \alpha_b$.

There are also defined on M the real connections ∇ and $\tilde{\nabla}$ of the real metrics h and k. It follows immediately from equation (4.1) that

$$D = \tfrac{1}{2}(\nabla + \tilde{\nabla}) \tag{4.2}$$

(as operators on holomorphic vectors and tensors) and hence that

$$\tfrac{1}{2}(\nabla + \tilde{\nabla})J = 0$$

[that is, $\frac{1}{2}(\nabla + \tilde{\nabla})$ is an *almost complex connection*]. Using this, and covariantly differentiating the compatibility condition $k(X, Y) = -k(JX, JY)$ with respect to ∇ and $\tilde{\nabla}$ for arbitrary vector fields X and Y, one deduces that

$$\nabla k = 0 \tag{4.3}$$

Hence the two metric connections ∇ and $\tilde{\nabla}$ coincide, and J is parallel with respect to both.

The converse is also true:

> *Proposition 1.* Let M be a real even-dimensional manifold on which there is defined a metric h and an almost complex structure J satisfying the compatibility condition $h(X, Y) = -h(JX, JY)$ $\forall X, Y$. Put $g = h + ik$, where k is defined by $k(X, Y) = -h(JX, Y)$. Then (M, g) is a complex Riemannian manifold if, and only if, J is parallel with respect to ∇ (the metric connection of h).

Proof. The "only if" part has already been established. To prove the converse, assume that $\nabla J = 0$. Then J is integrable, and hence M is complex (an almost complex structure that is parallel with respect to a torsion-free affine connection is necessarily integrable; see Kobayashi and Nomizu, 1969, p. 145). Thus, it only remains to be shown that the equation $\nabla J = 0$ also implies the Cauchy-Riemann equations for the components of g. These can be written in the form

$$\bar{X}(g(Y, Z)) = 0 \tag{4.4}$$

for all holomorphic vector fields X, Y and Z.

Let X and Y be holomorphic vector fields. Then $[X, \bar{Y}] = 0$ and hence $\nabla_X \bar{Y} = \nabla_{\bar{Y}} X$. Now, $JX = iX$ and $J\bar{Y} = -i\bar{Y}$. Thus, exploiting the hypothesis that J is parallel with respect to ∇;

$$-i(\nabla_X \bar{Y}) = J(\nabla_X \bar{Y}) = J(\nabla_{\bar{Y}} X) = i\nabla_{\bar{Y}} X \tag{4.5}$$

Therefore,

$$\nabla_X \bar{Y} = 0 = \nabla_{\bar{Y}} X \tag{4.6}$$

Hence, for any holomorphic vector fields X, Y, and Z,

$$\bar{X}(g(Y, Z)) = \nabla_{\bar{X}}(g(Y, Z)) = 0 \tag{4.7}$$

which completes the proof.

5. *Real Slices*

A *real slice* of a complex n-dimensional space-time (M, g) is an n-dimensional real analytic submanifold $N \subset M$ with $k|_N = 0$ and $h|_N$ nondegenerate. In other words, the metric g restricted to N is real and nondegenerate.

In particular, a real slice of (M, g) is a totally null (or isotropic) n-surface with respect to k; however (see also Rosca and Vanhecke, 1976), we have the following:

> *Proposition 2.* A totally null n-surface in a $2n$-dimensional pseudo-Riemannian manifold with signature $+, +, \cdots, -, -, \cdots$ (equal numbers of plus and minus signs) is necessarily totally geodesic (that is, any geodesic that touches the surface also lies in the surface).

Proof. Let (M, k) be a real pseudo-Riemannian manifold of dimension $2n$ and signature $+, +, \cdots, -, -, \cdots$, and let $N \subset M$ be a totally null n-surface [that is, $k(X, Y) = 0$ whenever X and Y are tangent to N]. Locally, N is given by the vanishing of n real functions $u, v, w, \ldots$. The vector fields $U^\alpha = \nabla^\alpha u$, $V^\alpha = \nabla^\alpha V$, etc. are normal to N, and hence also tangent to N since N is n-dimensional (here ∇ is the connection of the metric k, which is also used to raise and lower indices). Therefore, on N,

$$0 = U(k(V, W)) = U^\alpha V^\beta \nabla_\alpha W_\beta + U^\alpha W^\beta \nabla_\alpha V_\beta \tag{5.1}$$

$$0 = V(k(W, U)) = V^\alpha W^\beta \nabla_\alpha U_\beta + V^\alpha U^\beta \nabla_\alpha W_\beta \tag{5.2}$$

$$0 = W(k(U, V)) = W^\alpha U^\beta \nabla_\alpha V_\beta + W^\alpha V^\beta \nabla_\alpha U_\beta \tag{5.3}$$

Also, $\nabla_{[\alpha} U_{\beta]} = \nabla_{[\alpha} V_{\beta]} = \nabla_{[\alpha} W_{\beta]} = 0$. Hence, adding (5.1) and (5.2) and subtracting (5.3), we obtain

$$U^\alpha V^\beta \nabla_\alpha W_\beta = 0 \tag{5.4}$$

Thus, each $\nabla_U W$ is normal to N and hence also tangent to N. Finally, since any vector field X tangent to N can be written in the form

$$X = aU + bV + \cdots$$

for some functions $a, b, \ldots$, we conclude that $\nabla_X Y$ is tangent to N whenever X and Y are tangent to N and thus that N is totally geodesic (see Kobayashi and Nomizu, 1969, p. 56).

In particular, it follows that there is at most one real slice through any point of M with any given tangent plane, and that the intersection of two real slices is a totally geodesic submanifold of both (with respect to the induced metric connections).

Remarks. 1. A real slice N is necessarily an *analytic* submanifold of M since N is the image under the exponential map of the metric k or a real n-plane in the tangent space to M at any point of N, and the exponential map of a real analytic metric is real analytic. Further, N is also totally real since, if X and JX are tangent to N, then $h(X, Y) = -k(JX, Y) = 0$ for all Y tangent to N, implying $X = 0$ since $h|_N$ is nondegenerate. Hence a real slice is given locally by equation (2.5) in some complex coordinate system (see Shutrick, 1958). I am grateful to R. O. Wells for a discussion of this point.

2. A practical method of ruling out the existence of a real slice through a given point of M is to observe that if there is a real slice through $m \in M$ then its tangent plane will define a complex conjugation in the space of holomorphic tangents to M at m, and that the curvature tensor of g must be invariant under this conjugation. In the two component spinor notation of Penrose (1968), such a conjugation takes different forms according to the signature of the real slice. Thus, for a space-time real slice, the complex conjugate of a spinor α^A is of the form $\bar{\alpha}^{A'}$. In the positive definite case, it is $\bar{\alpha}_A$, and in the zero signature case, it is $\bar{\alpha}^A$. In particular, in the positive definite and zero signature cases, the conjugation does not interchange the self dual ($\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'}$) and anti-self dual ($\tilde{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}$) parts of the Weyl curvature. Thus, for example, in type (2,2) space-times, where

$$\Psi_{ABCD} = \psi_2 o_{(A} o_B \iota_C \iota_{D)} \text{ and } \tilde{\Psi}_{A'B'C'D'} = \tilde{\psi}_2 \tilde{o}_{(A'} \tilde{o}_{B'} \tilde{\iota}_{C'} \tilde{\iota}_{D')} \tag{5.6}$$

there can only be positive definite or zero signature slices through points where the scalars ψ_2 and $\tilde{\psi}_2$ are real.

6. *Isotropic 4-Planes*

Consider the various possibilities for the tangent plane to N at a fixed point $m \in M$. In the (real) tangent space $T_m M$, we have the metric k and the complex structure J. Since J defines an orientation for $T_m M$, there is also a natural volume element (8-form) ϵ.

Any 4-plane $P \subset T_m M$ that is isotropic with respect to k is spanned by four vectors A, B, C, D which are null and mutually orthogonal with respect to k. From these, we can construct a 4-form P with components

$$P^{\alpha\beta\gamma\delta} = A^{[\alpha} B^\beta C^\gamma D^{\delta]} \tag{6.1}$$

This form is automatically either self-dual or anti-self-dual with respect to ϵ and k. Thus there are two sorts of totally null 4-plane [these correspond to the two sorts of spinor for the group $O(4, 4)$; see Cartan (1966)],

α planes for which P is self-dual

β planes for which P is anti-self-dual

The condition that P should contain no (0, 1) vectors is the condition that A,B,C,D,JA,JB,JC, and JD should be linearly independent. In this case, it is possible to choose A, B, C, and D so that $k(A, JA) = \pm 1$, $k(A, JB) = 0$ and so on. It is then not hard to see that P is an α plane or a β plane as $k(A, JA)k(B, JB)k(C, JC)k(D, JD)$ is positive or negative. Thus, the possible signatures for h restricted to P are

α planes: $++++$, $++--$, $----$

β planes: $+++-$, $+---$

The intersection of two α planes or of two β planes is always even dimensional, while the intersection of an α plane with a β plane is always odd dimensional (Cartan, 1966, p. 108). Combining this with the results of Section 5, it follows, for example, that a positive definite real slice intersects a space-time real slice either in a single geodesic or in a totally geodesic spacelike hypersurface.

7. *Concluding Remarks*

Two consequences of all this are

(1) Twistor surfaces (totally null *complex* 2-surfaces in M) are the limiting case of real slices where $h|_N$ also vanishes. This may be important in the search for a method of passing from single graviton space-times (Penrose, 1976), which contain either α or β twistor surfaces according to helicity, to real space-times.

(2) Analytic continuation from space-times to Riemannian real slices is, in a sense, a more global procedure than, for example, continuation from $++++$ signature to $++--$ signature since the tangent planes of positive definite and space-time slices belong to different families.

Acknowledgments

I thank Roger Penrose and David Lerner for fruitful discussions.

Note Added in Proof

Some related ideas have been published by K. Rózga in *Reports on Mathematical Physics*, **11**, 197–210 (1977).

References

Cartan, E. (1966). *The Theory of Spinors*, M.I.T. Press, Cambridge, Massachusetts.
Flaherty, E. J. (1976). *Hermitian and Kahlerian Geometry in Rencontres*, deWitt, in Physics, Vol. 46, Springer-Verlag, Berlin.
Hartle, J. B., and Hawking, S. W. (1976). *Physical Review* D**13**, 2188.
Hawking, S. W., and Gibbons, G. W. (1977). *Physical Review* D**15**, 2738.
Kobayashi, S., and Nomizu, K. (1969). *Foundations of Differential Geometry*, Vol. 2, Wiley, New York.
Penrose, R. (1968). "Structure of Space-Time," in *Battelle Rencontres*, deWitt, C. M., and Wheeler, J. A., eds., Benjamin, New York.
Penrose, R. (1976). *General Relativity and Gravitation*, **7**, 31.
Plebanski, J. F., and Robinson, R. (1976). *Physical Review Letters*, **37**, 493.
Rosca, R., and Vanhecke, L. (1976). *Verhandelingen vande Koninklijke Academie voor Wetenschappen, Letteren en Schone Kunsten van Belgie, Klasse der Wetenschappe*, **38**, 1.
Shutrick, H. B. (1958). *Quarterly Journal of Mathematics*, Oxford (2) **9**, 189.